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CONTROL THEORY

On an Algorithm for Dynamic Reconstruction of the Input

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Abstract—We consider the problem of dynamic reconstruction of the input in a system described by a vector differential equation and nonlinear in the state variable. We indicate an algorithm that is stable under information noises and computational errors and is aimed at infinite system operation time. The algorithm is based on the dynamic regularization method.

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1. INTRODUCTION

We consider a dynamical system with a perturbation described by the ordinary differential equation

$$\dot{x}(t) = f(x(t)) + Cv(t), \qquad x(0) = x_0. \tag{1}$$

Here $t \in T = [0, +\infty)$ is the time variable, $x(t) \in \mathbb{R}^n$ and $v(t) \in \mathbb{R}^q$ are the state of the system and the dynamic perturbation, respectively, at time t, $x_0 \in \mathbb{R}^n$ is the initial state of the system, and C is an $n \times q$ matrix; the properties of the function f will be specified below. It can be a Lipschitz function or a continuously differentiable function.

The values $v(t)$ of the perturbation are not given in advance and satisfy the a priori constraint $v(t) \in Q$ $(t \geq 0)$, where Q is a given convex bounded closed set in $\mathbb{R}^q$. Any Lebesgue measurable function $v(\cdot) : [0, +\infty) \mapsto Q$ will be called an *admissible perturbation*.

The system trajectory $x(t)$ depends on the input $v = v(t)$. Neither the input nor the system trajectory is given in advance. In the course of motion, one observes a signal characterizing the state of the system. More precisely, the coordinates of system (1) are measured with error at discrete sufficiently frequent time instants $\tau_i \in T$ $(i = 0, 2, \ldots)$. The measurement results are vectors $\xi_i^h \in R^n$ and satisfy the conditions

$$|\xi_i^h - x(\tau_i)| \leq \nu_i^h, \tag{2}$$

where $\nu_i^h \in (0, 1)$ is the value of the error at time τ_i, the number $h \in (0, 1)$ specifies the measurement accuracy, and the symbol $|\cdot|$ stands for the Euclidean norm of a vector. The properties of ν_i^h will be specified in more detail below. Now we only note that $\nu_0^h = h$. Therefore,

$$|\xi_0^h - x_0| \leq h. \tag{3}$$

Our problem is to construct an algorithm for the approximate reconstruction of the unknown admissible perturbation $v(\cdot)$ on the basis of inexact measurements of the state $x(t)$ [i.e., $\xi(t)$, $t \geq 0$], which has the properties of being dynamic and stable. The algorithm being dynamic means that the approximations $v^h(t)$ to the current values $v(t)$ of the perturbation are constructed in real time. The stability property means that the approximation $v^h(t)$ is arbitrarily sharp if the accuracy of measurements is sufficiently high (the value of h is sufficiently small). The problem belongs to the class of problems of dynamic inversion for systems with perturbations; the desired algorithm can be classified as an algorithm of stable dynamic inversion (dynamic regularization).

The first algorithm of dynamic regularization suggested in [1] was a combination of the positional control method with a model [2] in guaranteed control theory and the Tikhonov regularization method [3] in the theory of ill-posed problems. Later, this algorithm was generalized to partially observable systems described by ordinary differential equations [4, 5] and to infinite-dimensional systems described by delay differential equations [6] and partial differential equations [7, 8]. (We mention only monographs and survey papers, where one can find additional references.)

It is important to note that algorithms of stable dynamic inversion suggested in the abovementioned papers were designed for approximating the perturbation on a bounded time interval $[0, \vartheta]$. As the length ϑ of this interval increases, the computational and measurement errors are accumulated; and as ϑ tends to infinity, the approximation performance is infinitely deteriorated. This performance is estimated by two criteria; first, by the value of the uniform (on $[0, \vartheta]$) deviation of the trajectory of system (1) corresponding to the true perturbation $v(\cdot)$ from the trajectory of some auxiliary system (referred to as a model) corresponding to the constructed approximation $v^h(\cdot)$ of that perturbation, and second, by the difference of the mean-square norms of the functions $v^h(\cdot)$ and $v(\cdot)$ (on $[0, \vartheta]$). The choice of these two criteria is explained by the fact that if they are small for an appropriate choice of the model (for the second criterion, only if it is positive), then the approximation $v^h(\cdot)$ is close to the perturbation $v(\cdot)$ in the mean-square norm on the interval $[0, \vartheta]$ provided that the matrix C has rank q.

In the present paper, we continue the research [4–8] and construct an algorithm for stable dynamic inversion of system (1) free of this disadvantage. Other algorithms combining elements of regularization and feedback control and aimed at infinite system operation time can be found in [9–11].

2. AUXILIARY CONSTRUCTIONS. STATEMENT OF THE PROBLEM

We assume that the solution $x(\cdot)$ of system (1) generated by an unknown admissible perturbation $v(\cdot)$ remains in a bounded domain $H^x \subset R^n$ for all $t \in T$; i.e., $x(t) = x(t; t_0, x_0, v(\cdot)) \in H^x$. Consequently,

$$|x(t)| \le H = \text{const} \in [0, +\infty) \quad \text{for all} \quad t \in T. \tag{4}$$

Let $(\Delta_h)_{h>0}$ be a fixed family of partitions of the half-line $[0, +\infty)$ by control time instants,

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{\infty}, \quad \tau_{h,0} = 0, \quad \tau_{h,i+1} = \tau_{h,i} + \delta_i(h), \quad \delta_i(h) \in (0, 1], \quad \sum_{i=0}^{+\infty} \delta_i(h) = +\infty.$$

Any piecewise constant function $\xi(\cdot): [0, +\infty) \mapsto \mathbb{R}^n$, $\xi^h(t) = \xi_i^h$ for $t \in [\tau_{h,i}, \tau_{h,i+1})$, satisfying the constraints (2) and (3) will be called an *admissible measurement of accuracy* h.

Consider two cases. In the first case, we assume that the noises implemented in the observation channel are subjected to the requirement of "smallness" of their mean values over the entire time interval of system operation (the "smallness" of their integral errors), and in the second case, they are subjected to the constraints of "smallness" of their values at each time. In addition, in the first case, we assume that the following condition is satisfied.

Condition 1. (a) The function $f(\cdot)$ satisfies the Lipschitz condition

$$|f(x_1) - f(x_2)| \le L|x_1 - x_2|, \quad x_1, x_2 \in \mathbb{R}^n. \tag{5}$$

(b) The family of partitions Δ_h and the values of the measurement errors ν_i^h satisfy the relations $\nu_i^h \in [0, 1]$ for all i and $h \in (0, 1)$ and

$$\sum_{i=0}^{+\infty} \delta_i(h)\nu_i^h \le \varphi_1(h) \to 0+ \quad \text{as} \quad h \to 0+\,.$$

In turn, in the second case, we assume that the following requirement is satisfied.

Condition 2. The relations $\delta_i(h) = \delta(h)$, $0 \le \nu_i^h \le h$, hold for all $i = 0, 1, \dots$

Before proceeding to the description of the reconstruction algorithm, we present some auxiliary constructions.

In the first case, we proceed as follows. We take two symmetric stable $n \times n$ matrices A and B; moreover, the first matrix has the form $A = -\gamma^{-1} I$ (I is the $n \times n$ identity matrix), where

$$\gamma = \text{const} \in \left(0, \frac{-\lambda_n}{n^{1/2} L \|B\|}\right). \tag{6}$$

Here and throughout the following, $\lambda_n < 0$ is the maximum eigenvalue of the matrix B, and the symbol $\|\cdot\|$ stands for the Euclidean norm of a matrix. Since B is a symmetric matrix, it follows that its eigenvalues are real; in addition, by [12, p. 39],

$$W(x) = x'Bx \le \lambda_n |x|^2 \qquad \forall x \in R^n. \tag{7}$$

It follows from Theorem 9.1 in [12, p. 35] that, for the matrices A and B, there exists a positive definite symmetric matrix D such that

$$\left(\frac{\partial V(x)}{\partial x}\right)' Ax = W(x) \qquad \forall x \in R^n, \tag{8}$$

where $V(x) = x'Dx$. Moreover, the matrix D satisfies the relation $B = 2DA$, which implies that

$$\|D\| \le \frac{1}{2}\|B\| \|A^{-1}\|.$$

In addition, $A^{-1} = -\gamma I$ and $\|A^{-1}\| = (n\gamma^2)^{1/2}$. Therefore [see (6)], we have

$$\|D\| \le \frac{1}{2}\|B\| n^{1/2}\gamma < \frac{|\lambda_n| n^{1/2}\|B\|}{2n^{1/2} L \|B\|} = \frac{|\lambda_n|}{2L}.$$

In this case,

$$\chi = 2L\|D\| + \lambda_n < 0. \tag{9}$$

Note that there exists a number $c_* > 0$ such that

$$\|e^{At}\| \le c_* e^{\lambda_n t} \qquad (t \ge 0). \tag{10}$$

In the second case, along with Condition 2, we assume that the following condition is satisfied (see [10]).

Condition 3. The function $f(x)$ is continuously differentiable, and $|\partial f(x)/\partial x| \le L_* < +\infty$ for any $x \in R^n$. There exists a positive definite matrix function D such that

$$2x' D \frac{\partial f(\bar{x})}{\partial x} x \le \chi |x|^2 \qquad \forall x, \bar{x} \in R^n,$$

where $\chi < 0$.

We introduce the auxiliary control system

$$\dot{y}^h(t) = Cv^h(t) + f_1(t, \xi^h(t), y^h(t)), \quad t \in T, \tag{11}$$

with initial state $y^h(0) = \xi_0^h$ and control $v^h(\cdot) \in Q(\cdot)$. Here

$$f_1(t, \xi^h(t), y^h(t)) = \begin{cases} f(y^h(t)) + A(y^h(t) - \xi^h(t)) & \text{in the first case} \\ f(y^h(t)) & \text{in the second case,} \end{cases}$$

and $Q(\cdot)$ is the set of Lebesgue measurable functions $v(\cdot):\ [0,+\infty)\to Q$. System (11) will be called a *modeling system* (a *model*).

For arbitrary admissible perturbations $v(\cdot)$ and $v^h(\cdot)$ [$v^h(\cdot)$ is treated as the constructed approximation to the unobservable perturbation $v(\cdot)$], we introduce two criteria of the deviation of $v^h(\cdot)$ from $v(\cdot)$ on some bounded time interval $[0,\vartheta]$:

$$\omega_1(v^h(\cdot),v(\cdot)|\vartheta)=\max_{t\in[0,\vartheta]}|y^h(t;t_0,\xi_0^h,v^h(\cdot))-x(t;t_0,x_0,v(\cdot))|,\tag{12}$$

$$\omega_2(v^h(\cdot),v(\cdot),h|\vartheta)=\int\limits_0^\vartheta|v^h(t)|^2\,dt-\varrho_1(h)\int\limits_0^\vartheta|v(t)|^2\,dt,\quad \varrho_1(h)\to 1\quad\text{as}\quad h\to 0.\tag{13}$$

Here $x(\cdot\,;t_0,x_0,v(\cdot))$ and $y^h(\cdot\,;t_0,\xi_0^h,v^h(\cdot))$ stand for trajectories of systems (1) and (11) induced by the inputs $v(\cdot)$ and $v^h(\cdot)$, respectively.

We assume that the states $y^h(t)$, $t\geq 0$, of the model (11) are observed at discrete time instants $\tau_{h,i}$ with an error and change under the action of a specially formed feedback $V(t,y^h(\cdot),\xi^h(\cdot))\in Q$, which "simulates" the values $v(t)$ of the unobservable perturbation in system (11). Therefore, the motion of the model (11) depends on the results $\xi^h(\cdot)$ of the measurement of the trajectory of system (1) and satisfies the following differential equation and initial condition:

$$\dot y^h(t)=f_1(t,\xi^h(t),y^h(t))+CV(\tau_i,\xi_i^h,\psi_i^h),\qquad t\in\delta_i,\qquad y^h(0)=\xi_0^h,\tag{14}$$

where ψ_i^h are the results of inexact measurements of the state $y^h(\tau_i)$, i.e., $|\psi_i^h-y^h(\tau_i)|\leq\nu_i^h$. Any function

$$V(\cdot\,,\cdot\,,\cdot):\ T\times\mathbb{R}^n\times\mathbb{R}^n\mapsto Q$$

is referred to as an *admissible feedback* [for the model (11)]. Obviously, for any admissible feedback $V(\cdot\,,\cdot\,,\cdot)$ and for any admissible measurement $\xi^h(\cdot)$ of accuracy h, there exists a solution $y^h(\cdot)=y^h(\cdot\,;t_0,\xi_0^h,v^h(\cdot))$ of the Cauchy problem (14) defined on $[0,+\infty)$, which is referred to as the trajectory of the model corresponding to the admissible feedback $V(\cdot\,,\cdot\,,\cdot)$ and the admissible measurement $\xi^h(\cdot)$.

The *control process* corresponding to an admissible feedback $V(\cdot\,,\cdot\,,\cdot)$, an admissible perturbation $v(\cdot)$, and the measurement accuracy h ($h>0$) is defined as an arbitrary quadruple $(x(\cdot),\xi^h(\cdot),$ $y^h(\cdot),v^h(\cdot))$, where $x(\cdot)=x(\cdot\,;t_0,x_0,v(\cdot))$ is the trajectory of system (1), $\xi^h(\cdot)$ is an admissible measurement of accuracy h corresponding to $x(\cdot)$, $y^h(\cdot)$ is the trajectory of the model (14) corresponding to $V(\cdot\,,\cdot\,,\cdot)$ and $\xi^h(\cdot)$, and the function $v^h(\cdot):\ [0,+\infty)\mapsto Q$ has the form

$$v^h(t)=V(\tau_i,\xi_i^h,\psi_i^h),\qquad t\in\delta_i=[\tau_i,\tau_{i+1}),\qquad \tau_i=\tau_{h,i},$$

where $|\psi_i^h-y^h(\tau_i)|\leq\nu_i^h$ and $\xi_i^h=\xi^h(\tau_i)$. In this case, the function $v^h(\cdot)$ is referred to as an *implementation of the admissible strategy* $V(\cdot\,,\cdot\,,\cdot)$ [corresponding to the admissible perturbation $v(\cdot)$] and an admissible measurement accuracy h.

To solve the considered problem, we use the family $(V_h(\cdot\,,\cdot\,,\cdot))_{h>0}$ of admissible feedbacks. Following [10], we say that it is stable with respect to time ϑ if there exist functions $\gamma_1(\cdot),\gamma_2(\cdot)$: $(0,+\infty)\mapsto[0,+\infty)$ such that $\gamma_1(h),\gamma_2(h)\to 0$ as $h\to 0$ and the inequalities [see (12) and (13)]

$$\sup_{\vartheta\geq 0}\omega_1(v^h(\cdot),v(\cdot)|\vartheta)\leq\gamma_1(h),\qquad \sup_{\vartheta\geq 0}\omega_2(v^h(\cdot),v(\cdot),h|\vartheta)\leq\gamma_2(h)$$

hold for any admissible perturbation $v(\cdot)$, any $h>0$, any implementation $v^h(\cdot)$ of the admissible feedback $V_h(\cdot\,,\cdot\,,\cdot)$,

$$v^h(t)=V_h(\tau_{h,i},\xi_i^h,\psi_i^h),\qquad t\in\delta_{h,i}=[\tau_{h,i},\tau_{h,i+1}),\tag{15}$$

any trajectory $y^h(t)=y^h(t;t_0,\xi_0^h,v^h(\cdot))$ of the model (14) corresponding to the perturbation $v^h(\cdot)$ of the form (15), and any admissible measurement $\xi^h(\cdot)$ of accuracy h.

In this case, the pair $(\gamma_1(\cdot), \gamma_2(\cdot))$ is referred to as an *accuracy estimate of the family* $(V_h(\cdot\,,\cdot\,,\cdot))_{h>0}$.

The considered problem of stable inversion of system (1) is to construct a family of admissible feedbacks V_h stable with respect to time ϑ.

Let us outline the reasons for choosing the functions $\omega_1(\cdot)$ and $\omega_2(\cdot)$ of the form (12) and (13), respectively, as criteria of the deviation of $\bar{v}(\cdot)$ from $v(\cdot)$ on the closed interval $[0, \vartheta]$. Let $\Omega_\vartheta(x(\cdot))$ stand for the set of all perturbations

$$v(\cdot) \in Q_\vartheta(\cdot) = \{v(\cdot) \in L_2([0, \vartheta]; \mathbb{R}^q) :\ v(t) \in Q \ \text{ for almost all } \ t \in [0, \vartheta]\}$$

that generate the trajectory $x(t)$, $t \in [0, \vartheta]$, of system (1); i.e.,

$$\Omega_\vartheta(x(\cdot)) = \{v(\cdot) \in Q_\vartheta(\cdot) :\ x(t; t_0, x_0, v(\cdot)) = x(t),\ t \in [0, \vartheta]\}.$$

By $v_\vartheta(\cdot)$ we denote the minimum (in the sense of the norm of the space $L_2([0, \vartheta]; \mathbb{R}^q)$) perturbation in the set $\Omega_\vartheta(x(\cdot))$. Since Q is a convex compact set, it follows that such a perturbation is unique. Suppose that $\{h_j\}$ are chosen numerical sequences, $h_j \to +0$ as $j \to +\infty$, $\{\bar{v}_\vartheta^{h_j}(\cdot)\} \in Q_\vartheta(\cdot)$, and $\bar{v}_\vartheta^{h_j}(\cdot)$ converges weakly to some perturbation in $L_2([0, \vartheta]; \mathbb{R}^q)$ as $j \to +\infty$. Then, as follows from the results in [1, 4], the relations

$$\omega_1(\bar{v}_\vartheta^{h_j}(\cdot), v_\vartheta(\cdot)|\vartheta) \le \gamma_1(h_j), \qquad \omega_2(\bar{v}_\vartheta^{h_j}(\cdot), v_\vartheta(\cdot)|\vartheta) \le \gamma_2(h_j)$$

lead to the strong convergence of $\bar{v}_\vartheta^{h_j}(\cdot)$ to $v_\vartheta(\cdot)$ in $L_2([0, \vartheta]; \mathbb{R}^q)$ as $j \to +\infty$. But if the matrix C has rank q and $v(t) = C^{-1}(\dot{x}(t) - f(x(t))) \in Q$, $t \in [0, \vartheta]$, then $\bar{v}_\vartheta^{h_j}(\cdot)$ converges to the true control $v_\vartheta(\cdot)$ applied to the system (1).

3. ALGORITHM FOR SOLVING THE PROBLEM

Before proceeding to the description of the algorithm for solving the considered problem, we introduce the following conditions.

Condition 4. The inequality $\inf\{|u| :\ u \in Q\} \ge 1$ holds.

Condition 5. The family of partitions Δ_h satisfies the inequality

$$\sum_{i=0}^{+\infty} \delta_i^2(h) \le \varphi_2(h), \quad \varphi_2(h) \to 0 \quad \text{as} \quad h \to 0.$$

Remark 1. Conditions 1(b) and 5 are satisfied if, for example,

$$\delta_i(h) = \nu_i^h = dh/(i+1)^\mu \le 1, \quad \mu \in (0.5; 1], \quad i = 0, 1, \ldots, \quad d = \text{const} > 0.$$

In this case,

$$\varphi_1(h) = \varphi_2(h) = 2h^2 d^2 \sum_{i=1}^{\infty} i^{-2\mu},$$

and inequality (2) acquires the form

$$|\xi_i^h - x(\tau_i)| \le dh/(i+1)^\mu.$$

Before the operation of the algorithm, we fix the quantity h, the family $\{\nu_i^h\}_{i=0}^\infty$, the function $\alpha = \alpha(h)$, and the partition $\Delta_h = \{\tau_{h,i}\}_{i=0}^\infty$. We split the algorithm into similar steps. During the ith step performed on the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, one makes the following operations. First, at time τ_i, the vector $V_h(\tau_{h,i}, \xi_i^h, \psi_i^h)$ is evaluated by the formula

$$V_h(\tau_{h,i}, \xi_i^h, \psi_i^h) = \arg\min\{2(D(\psi_i^h - \xi_i^h))' C v + \alpha |v|^2 :\ v \in Q\}. \tag{16}$$

Then the control (15) is supplied to the input of the model (11) during the time interval. As a result, under the action of that control and some unknown perturbation $v(t)$, $t \in \delta_i$, system (1) passes from the state $x(\tau_i)$ into the state $x(\tau_{i+1})$, and the model (11) passes from the state $y^h(\tau_i)$ into the state $y^h(\tau_{i+1})$. Similar actions are repeated at the $(i+1)$st step.

Let us show that, for an appropriate choice of the function $\alpha(\cdot)$, the family $(V_h(\cdot\,,\cdot\,,\cdot))_{h>0}$ is the desired family of admissible feedbacks stable with respect to time ϑ.

Theorem 1. *Let Conditions* 1 *and* 5 (*in the first case*) *or Conditions* 2–4 (*in the second case*) *be satisfied, let* $h \in (0,1)$, *and let* $(x(\cdot), \xi^h(\cdot), y^h(\cdot), v^h(\cdot))$ *be a control process corresponding to an admissible feedback* $V_h(\cdot\,,\cdot\,,\cdot)$, *an admissible perturbation* $v(\cdot)$, *and an admissible measurement* $\xi^h(\cdot)$ *of accuracy* h. *Then the inequality*

$$\int\limits_0^t |v^h(\tau)|^2\,d\tau - \varrho_1(h)\int\limits_0^t |v(\tau)|^2\,d\tau \le \varrho_2(h) \tag{17}$$

holds for all $t \ge 0$, *where*

$$\varrho_1(h) = 1, \qquad \varrho_2(h) = \frac{|(\xi_0^h - x_0)' D(\xi_0^h - x_0)| + b_0(\varphi_1(h) + \varphi_2(h))}{\alpha(h)}$$

in the first case and

$$\varrho_1(h) = \frac{\alpha(h) + b_1(h + \delta(h))}{\alpha(h) - b_1(h + \delta(h))}, \qquad \varrho_2(h) = \frac{|(\xi_0^h - x_0)' D(\xi_0^h - x_0)|}{\alpha(h) - b_1(h + \delta(h))}$$

in the second case.

In addition, the inequalities

$$|y^h(t) - x(t)| \le \nu(h,\alpha) \tag{18}$$

hold for all $t \ge 0$, *where*

$$\nu(h,\alpha) = \chi_*^{-1/2}\left[|(\xi_0^h - x_0)' D(\xi_0^h - x_0)| e^{\chi_1 t} - 2c_0^2\frac{\alpha(h)}{\chi_1} + 2b_0(\varphi_1(h) + \varphi_2(h))\right]^{1/2}$$

in the first case and

$$\nu(h,\alpha) = \chi_*^{-1/2}\left[|(\xi_0^h - x_0)' D(\xi_0^h - x_0)| e^{\chi_1 t} - \frac{2c_0^2\alpha(h) + 2c_0 b_1(h + \delta(h))}{\chi_1}\right]^{1/2}$$

in the second case.

Here

$$\begin{gathered}
c_0 = \max_{v\in Q}|v|, \qquad c_1 = 4\|D\|\max_{v\in Q}|Cv|, \qquad c_2 = 2\max_{v\in Q}|Cv|,\\
d_0 = |f(0)| + LH + \max_{v\in Q}|Cv|, \qquad d_1 = (L + \|A\|)\varkappa + 2\max_{v\in Q}|Cv| + \|A\|(1 + d_0),\\
d_2 = 2\|D\|\,\|A\|\max\{1, d_0\}\varkappa, \qquad d_3 = c_2 - \frac{c_1 L_*\chi_1}{\chi_*^2},\\
\varkappa = c_*(1 - 2\max_{v\in Q}|Cv| + \|A\|(1 + d_0)\lambda_n^{-1})\exp(-c_0 L\lambda_n^{-1}),\\
b_0 = c_1(2 + d_1) + d_2, \qquad b_1 = 2\|D\|\,\|C\|\max\{2, d_3\},
\end{gathered} \tag{19}$$

c_* *and* χ *are defined in* (9) *and* (10), *respectively*, $\chi_1 = \chi_0\chi$, *and* $\chi_0 > 0$ *and* $\chi_* > 0$ *are numbers such that*

$$\chi_*|x|^2 \le |x'Dx|, \qquad \chi_0|x'Dx| \le |x|^2 \qquad \forall x \in R^n. \tag{20}$$

Remark 2. Under the assumptions of Theorem 1, if either

$$(h + \varphi_1(h) + \varphi_2(h))/\alpha(h) \to 0$$

(in the first case) or

$$(h + \delta(h))/\alpha(h) \to 0$$

(in the second case) as $h \to 0$, and the above-mentioned measurement $\xi(\cdot)$ [corresponding to an admissible perturbation $v(\cdot)$] has accuracy h, i.e., inequalities (2) and (3) hold, then for a small h, the right-hand sides of inequalities (17) and (18) are small uniformly with respect to $t \geq 0$. Consequently, the admissible feedback $V_h(\cdot\,,\cdot)$ provides a solution of the problem on the approximate shadowing of the inexactly measured motion of system (1) on the half-open time interval $[0, +\infty)$ by the controlled model (14) under the additional condition that, on any finite time interval $[0, t]$, the mean-square norm of the control $v^h(\cdot)$ of the model does not (approximately) exceed the mean-square norm of the unobservable perturbation $v(\cdot)$ applied to the system.

Proof of Theorem 1. Consider the first case. We have [see (11)]

$$\dot{y}^h(t) = A(y^h(t) - \xi^h(t)) + f(y^h(t)) + Cv^h(t) \quad (\text{almost all} \quad t \geq 0), \quad y(0) = \xi_0^h.$$

This, together with Eq. (1), implies that the difference

$$z(\cdot) = y^h(\cdot) - x(\cdot)$$

satisfies the relation

$$\dot{z}(t) = f(y^h(t)) - f(x(t)) + A(y^h(t) - \xi^h(t)) + C(v^h(t) - v(t)) \quad \text{for almost all} \quad t \geq 0, \tag{21}$$

where

$$z(0) = \xi_0^h - x_0.$$

In turn, system (21) can be represented in the form

$$\dot{z}(t) = Az(t) + C(v^h(t) - v(t)) + f(y^h(t)) - f(x(t)) + \psi_h(t) \quad \text{for almost all} \quad t \in T. \tag{22}$$

Here $\psi_h(t) = A(x(t) - \xi^h(t))$.

Let us show that $z(\cdot)$ is a bounded function. By taking into account the Cauchy formula and inequality (10), for $z(\cdot)$ with an arbitrary admissible perturbation $v(\cdot)$ and an arbitrary $t \geq 0$, we have

$$\begin{aligned} |z(t)| &\leq |e^{At}z(0)| + \int_0^t \|e^{A(t-\tau)}\|\,|f(x(\tau)) - f(y^h(\tau)) + C(v(\tau) - v^h(\tau)) + \psi_h(\tau)|\,d\tau \\ &\leq c_* e^{\lambda_n t}|z(0)| + c_* \int_0^t e^{\lambda_n(t-\tau)}\{c_2 + L|z(\tau)| + |\psi_h(\tau)|\}\,d\tau, \end{aligned} \tag{23}$$

where L is the Lipschitz constant of the function f [see (5)]. Note that, by virtue of inequalities (4) and (5), we have the estimate

$$|\dot{x}(t)| = |f(x(t)) + Cv(t)| \leq |f(0)| + L|x(t)| + |Cv(t)| \leq d_0. \tag{24}$$

By taking into account the inequalities $\nu_i^h \leq 1$ (see Condition 1), $\delta_i(h) \leq 1$, and relation (24), for $t \in \delta_{h,i} = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, we have

$$\begin{aligned} |\psi_h(t)| = |A(\xi^h(t) - x(t))| &\leq \|A\|\left(|\xi_i^h - x(\tau_{h,i})| + \int_{\tau_{h,i}}^{\tau_{h,i+1}} |\dot{x}(\tau)|\,d\tau\right) \\ &\leq \|A\|(\nu_i^h + d_0\delta_i(h)) \leq c_3 = \|A\|(1 + d_0). \end{aligned} \tag{25}$$

Therefore, by virtue of the estimate (25), inequality (23) acquires the form

$$|z(t)| \le c_* e^{\lambda_n t}|z(0)| + c_* \int\limits_0^1 e^{\lambda_n(t-\tau)}\{c_2 + c_3 + L|z(\tau)|\}\,d\tau.$$

By using this estimate and the estimates (10), (3), and

$$\int\limits_0^t e^{\lambda_n(t-\tau)}\,d\tau \le -\lambda_n^{-1}, \qquad |z(0)| \le h \le 1, \tag{26}$$

we obtain the inequality

$$|z(t)| \le c_4 + \int\limits_0^t c_* L e^{\lambda_n(t-\tau)}|z(\tau)|\,d\tau,$$

where $c_4 = c_*(1 - (c_2 + c_3)\lambda_n^{-1})$. By the Gronwall lemma, from the last inequality, we derive the estimate

$$|z(t)| \le c_4 \exp\left(\int\limits_0^t c_* L e^{\lambda_n(t-\tau)}\,d\tau\right) \le \varkappa = c_4 e^{-c_* L \lambda_n^{-1}}. \tag{27}$$

By $\dot V(t)|_{(22)}$ we denote the derivative of the Lyapunov function $V(x) = x'Dx$ according to system (22); i.e.,

$$\dot V(t)|_{(22)} = \left(\left.\frac{\partial V}{\partial x}\right|_{x=z(t)}\right)'\{Az(t) + C(v^h(t) - v(t)) + f(y^h(t)) - f(x(t))\} + \Phi(t) \quad \text{for almost all} \quad t \in T,$$

where

$$\Phi(t) = \left(\left.\frac{\partial V}{\partial x}\right|_{x=z(t)}\right)' A(x(t) - \xi^h(t)).$$

By using relations (7) and (8), we obtain

$$\begin{aligned}\dot V(t)|_{(22)} \le \lambda_n|z(t)|^2 + \left(\left.\frac{\partial V}{\partial x}\right|_{x=z(t)}\right)' &\{C(v^h(t) - v(t)) + f(y^h(t)) - f(x(t))\} \\ &+ |\Phi(t)| \quad \text{for almost all} \quad t \in T.\end{aligned} \tag{28}$$

Next, by virtue of the Lipschitz property of the function f [see (5)], we have the inequality

$$\begin{aligned}&\left(\left.\frac{\partial V}{\partial x}\right|_{x=z(t)}\right)'(f(y^h(t)) - f(x(t))) \\ &\qquad \le |2(Dz(t))'(f(y^h(t)) - f(x(t)))| \le 2L\|D\|\,|z(t)|^2 \quad \text{for almost all} \quad t \in T.\end{aligned} \tag{29}$$

In turn, from (27), we derive the estimate

$$\left|\left.\frac{\partial V}{\partial x}\right|_{x=z(t)}\right| \le c_5 \quad \text{for} \quad t \in T,$$

where $c_5 = 2\|D\|\varkappa$. Consequently, by taking into account the estimate (25), we obtain

$$|\Phi(t)| \le d_2 \nu^h(t). \tag{30}$$

Here d_2 is the number defined in (19), and

$$\nu^h(t) = \nu_i^h + \delta_i(h) \quad \text{for almost all} \quad t \in \delta_{h,i}. \tag{31}$$

From (28)–(31), we have the estimate

$$\dot V(t)|_{(22)} \le 2(Dz(t))'C(v^h(t) - v(t)) + \chi|z(t)|^2 + d_2\nu^h(t) \quad \text{for almost all} \quad t \in T, \tag{32}$$

where χ is the number defined in (9).

Set

$$\varepsilon(t) = V(z(t)) + \alpha(h)\left[\int_0^t |v^h(\tau)|^2\,d\tau - \int_0^t |v(\tau)|^2\,d\tau\right] \qquad (t \ge 0). \tag{33}$$

By virtue of relations (22), (25), and (27), we have the inequality

$$|\dot z(t)| \le (L + \|A\|)|z(t)| + c_2 + |\psi_h(t)| \le d_1.$$

Therefore,

$$|z(t) - z(\tau_i)| \le d_1(t - \tau_i), \qquad t \in \delta_i = \delta_{i,h} = [\tau_i, \tau_{i+1}), \qquad \tau_i = \tau_{i,h}. \tag{34}$$

By taking into account inequality (34) and (24) and definition (19), we have

$$2(D(z(t) - z(\tau_i)))'C(v^h(t) - v(t)) \le c_1(t - \tau_i)d_1 \le c_1 d_1 \delta_i(h), \tag{35}$$
$$2(D(\xi_i^h - x(\tau_i)))'C(v^h(t) - v(t)) \le c_1\nu_i^h, \tag{36}$$
$$2(D(y^h(\tau_i) - \psi_i^h))'C(v^h(t) - v(t)) \le c_1\nu_i^h. \tag{37}$$

From relations (32), (33), and (35), we find that the inequality

$$\begin{aligned}\dot\varepsilon(t) \le\ & 2(D(\psi_i^h - \xi_i^h))'C(v^h(t) - v(t)) + \alpha(h)\{|v^h(\tau)|^2 - |v(\tau)|^2\}\\ &+ 2(D(y^h(\tau_i) - \psi_i^h))'C(v^h(t) - v(t)) + 2(D(\xi_i^h - x(\tau_i)))'C(v^h(t) - v(t))\\ &+ d_2\nu^h(t) + \chi|z(t)|^2 + c_1 d_1\delta_i(h)\end{aligned} \tag{38}$$

holds for almost all $t \in \delta_i$. In turn, it follows from relations (15) and (16) that the sum of the first two terms on the right-hand side in the last inequality is nonpositive. Therefore, by virtue of (36)–(38), the inequality

$$\dot\varepsilon(t) \le b_0\nu^h(t) + \chi|z(t)|^2 \tag{39}$$

holds for almost all $t \in \delta_i$, where $b_0 = d_2 + 2c_1 + c_1 d_1$ [see (19)]. Consequently, by taking into account the inequality $\chi < 0$ [see (9)] and Conditions 1 and 5, from inequality (39), we obtain

$$\varepsilon(t) \le \varepsilon(0) + b_0(\varphi_1(h) + \varphi_2(h)) \le |(\xi_0^h - x_0)'D(\xi_0^h - x_0)| + b_0(\varphi_1(h) + \varphi_2(h)) \qquad (t \ge 0).$$

This implies inequality (17) (in the first case).

Now let us show that inequality (18) is also true. By taking into account inequalities (32), (34), and (35), by analogy with (38), we obtain the estimate

$$\begin{aligned}\dot V(t)|_{(22)} \le\ & 2(D(\psi_i^h - \xi_i^h))'C(v^h(t) - v(t)) + 2(D(y^h(\tau_i) - \psi_i^h))'C(v^h(t) - v(t))\\ &+ 2(D(\xi_i^h - x(\tau_i)))'C(v^h(t) - v(t)) + d_2\nu^h(t)\\ &+ \chi|z(t)|^2 + c_1 d_1\delta_i(h) \quad \text{for almost all} \quad t \in \delta_i.\end{aligned} \tag{40}$$

In turn, it follows from (16) that

$$2(D(\psi_i^h - \xi_i^h))'Cv^h(t) \le \min\{2(D(\psi_i^h - \xi_i^h))'Cv(t) :\ v \in Q\} + 2c_0^2\alpha(h). \tag{41}$$

Therefore, the first term on the right-hand side in the estimate (40) does not exceed $2c_0^2\alpha(h)$. As was mentioned above [see (36) and (37)], the sum of the second and third terms on the right-hand side in (40) can be estimated from above by $2c_1\nu_i^h$. Therefore, for almost all $t \geq 0$, we obtain

$$\dot{V}(t)|_{(22)} \equiv \frac{d}{dt}|z'(t)Dz(t)| \leq \chi|z(t)|^2 + 2c_0^2\alpha(h) + (2c_1 + c_1d_1 + d_2)\nu^h(t),$$

where $\nu^h(t)$ is the function defined by relation (31) and $\chi < 0$ is the number defined in (9). Since [see (20)]

$$\chi|z(t)|^2 \leq \chi_1|z'(t)Dz(t)| \qquad (t \geq 0),$$

we have

$$\frac{d}{dt}|z'(t)Dz(t)| = \chi_1|z'(t)Dz(t)| + 2c_0^2\alpha(h) + (2c_1 + c_1d_1 + d_2)\nu^h(t) + \Psi(t) \quad (\text{for almost all} \quad t \geq 0),$$

where $\Psi(t) \leq 0$ $(t \geq 0)$. Hence for an arbitrary $t \geq 0$, we obtain the inequality

$$|z'(t)Dz(t)| \leq |z'(0)Dz(0)|e^{\chi_1 t} + 2c_0^2\int_0^t e^{\chi_1(t-\tau)}\alpha(h)\,d\tau + (2c_1 + c_1d_1 + d_2)\int_0^t e^{\chi_1(t-\tau)}\nu^h(\tau)\,d\tau. \tag{42}$$

By noting that

$$\int_0^t e^{\chi_1(t-\tau)}\alpha(h)\,d\tau \leq -\frac{\alpha(h)}{\chi_1} \qquad \forall t > 0$$

and by integrating by parts, we obtain

$$\int_0^t e^{\chi_1(t-\tau)}\nu^h(\tau)\,d\tau \leq \beta(t) - \chi_1\int_0^t e^{\chi_1(t-\tau)}\beta(\tau)\,d\tau, \tag{43}$$

where

$$\beta(\tau) = \int_0^\tau \nu^h(s)\,ds.$$

By virtue of Conditions 1 and 5, we have

$$\beta(\tau) \leq \int_0^{+\infty} \nu^h(s)\,ds = \sum_{i=0}^{+\infty}\delta_i(h)\{\nu_i^h + \delta_i(h)\} \leq \varphi_1(h) + \varphi_2(h) \qquad (\tau \geq 0).$$

Therefore, the right-hand side of inequality (43) does not exceed $2(\varphi_1(h) + \varphi_2(h))$. By (43), inequality (42) acquires the form

$$|z'(t)Dz(t)| \leq |z'(0)Dz(0)|e^{\chi_1 t} - 2c_0^2\frac{\alpha(h)}{\chi_1} + 2(2c_1 + c_1d_1 + d_2)(\varphi_1(h) + \varphi_2(h)),$$

which, together with (19), implies the inequality

$$|z(t)|^2 \leq \chi_*^{-1}\left(|z'(0)Dz(0)|e^{\chi_1 t} - 2c_0^2\frac{\alpha(h)}{\chi_1} + 2(2c_1 + c_1d_1 + d_2)(\varphi_1(h) + \varphi_2(h))\right).$$

We have thereby proved inequality (18) (in the first case). This completes the proof of the theorem in the first case.

Consider the second case. First, we show that $z(\cdot)$ is a bounded function. As was mentioned in [10], there exists a measurable function

$$\eta(t) \in \{\lambda x(t) + (1-\lambda) y^h(t) : \ \lambda \in [0,1]\}$$

such that

$$\dot{z}(t) = \frac{\partial f}{\partial x}(\eta(t)) z(t) + C(v^h(t) - v(t)) \quad \text{for almost all} \quad t \geq 0. \tag{44}$$

Then

$$\begin{aligned} \dot{V}(t)|_{(44)} &= \frac{d}{dt} V(z(t)) = 2(Dz(t))' \frac{\partial f}{\partial x}(\eta(t)) z(t) + 2(Dz(t))' C(v^h(t) - v(t)) \\ &\leq \chi |z(t)|^2 + c_1 |z(t)| \leq \chi_1 V(z(t)) + c_1 \chi_*^{-1/2} V^{1/2}(z(t)), \end{aligned} \tag{45}$$

where $V(z) = z' D z$ and D is a symmetric positive definite matrix (see Condition 3).

Let $\zeta_0(\cdot)$ be an upper solution of the equation

$$\dot{\zeta}_0(t) = \chi_1 \zeta_0(t) + c_1 \chi_*^{-1} \zeta_0^{1/2}(t), \qquad \zeta_0(0) = 0,$$

on $[0, \infty)$. This, together with (44), implies the estimate

$$\chi_* |z(t)|^2 \leq V(z(t)) \leq \zeta_0(t) \tag{46}$$

for all $t \geq 0$. By dividing the right- and left-hand sides of the last relation by $2\zeta_0^{1/2}(t)$ and by setting

$$\zeta_1(\cdot) = \zeta_0^{1/2}(\cdot), \tag{47}$$

we obtain

$$\dot{\zeta}_1(t) = \chi_1 \zeta_1(t)/2 + c_1 \chi_*^{-1}/2, \qquad \zeta_1(0) = 0.$$

By virtue of (26), the inequality

$$\zeta_1(t) = \frac{1}{2} c_1 \chi_*^{-1} \int\limits_0^t \exp\left(\frac{1}{2}\chi_1 (t - \tau)\right) d\tau \leq -\frac{c_1 \chi_1}{\chi_*}$$

holds for $t \in T$. Therefore, by taking into account relations (46) and (47), we obtain the estimate

$$|z(t)| \leq K_1 = -\frac{c_1 \chi_1}{\chi_*^2}. \tag{48}$$

From Condition 3, relation (44), and the estimate (48), we have

$$|\dot{z}(t)| \leq L_* |z(t)| + c_2 \leq d_3 \quad \text{for almost all} \quad t \in T.$$

Therefore,

$$|z(t) - z(\tau_i)| \leq d_3 (t - \tau_i) \quad \text{for} \quad t \in \delta_i = \delta_{h,i} = [\tau_i, \tau_{i+1}), \quad \tau_i = \tau_{h,i}. \tag{49}$$

Note that

$$\dot{\varepsilon}(t) = \dot{V}(t)|_{(44)} + \alpha(h)\{|v^h(t)|^2 - |v(t)|^2\}.$$

By using this relation, inequality (49), and the estimates

$$\begin{aligned} &2(D(z(t) - z(\tau_i)))' C(v^h(t) - v(t)) \leq 2\|D\| \, \|C\| d_3 \delta(h) \{|v^h(t)| + |v(t)|\}, \\ &2(D(\xi_i^h - x(\tau_i)))' C(v^h(t) - v(t)) \leq 2\|D\| \, \|C\| h \{|v^h(t)| + |v(t)|\}, \\ &2(D(\xi_i^h - x(\tau_i)))' C(v^h(t) - v(t)) \leq 2\|D\| \, \|C\| h \{|v^h(t)| + |v(t)|\} \quad \text{for almost all} \quad t \in \delta_i, \end{aligned} \tag{50}$$

we obtain

$$\dot{\varepsilon}(t) \le 2(D(\psi_i^h - \xi_i^h))'C(v^h(t) - v(t)) + \alpha(h)\{|v^h(t)|^2 - |v(t)|^2\} \\ + b_1(h + \delta(h))\{|v^h(t)| + |v(t)|\} + \chi|z(t)|^2 \quad \text{for almost all} \quad t \in \delta_i, \tag{51}$$

where b_1 is the number defined in (19) and $\varepsilon(t)$ is the function defined in (33). By using the definition of the control $v^h(\cdot)$ [see relations (16) and (15)], we obtain

$$\dot{\varepsilon}(t) \le b_1(h + \delta(h))\{|v^h(t)| + |v(t)|\} + \chi|z(t)|^2 \quad \text{for almost all} \quad t \in \delta_i.$$

Therefore, for $t \in [\tau_i, \tau_{i+1})$, we have

$$\varepsilon(t) \le \varepsilon(\tau_i) + b_1(h + \delta(h)) \int\limits_{\tau_i}^{t} \{|v^h(t)| + |v(t)|\}\, dt,$$

whence we obtain the inequality

$$\varepsilon(t) \le \varepsilon(0) + b_1(h + \delta(h)) \int\limits_0^t \{|v^h(\tau)|^2 + |v(\tau)|^2\}\, d\tau, \qquad t \in T. \tag{52}$$

In turn, from (52) for $t \in T$, we have

$$\alpha(h) \int\limits_0^t \{|v^h(\tau)|^2 - |v(\tau)|^2\}\, d\tau \le \varepsilon(0) + b_1(h + \delta(h)) \int\limits_0^t \{|v^h(\tau)|^2 + |v(\tau)|^2\}\, d\tau.$$

Consequently, the estimate

$$\{\alpha(h) - b_1(h + \delta(h))\} \int\limits_0^t |v^h(\tau)|^2\, d\tau \le \varepsilon(0) + \{\alpha(h) + b_1(h + \delta(h))\} \int\limits_0^t |v(\tau)|^2\, d\tau \tag{53}$$

holds for all $t \in T$. Hence, we derive the inequality

$$\int\limits_0^t |v^h(\tau)|^2\, d\tau \le \frac{\alpha(h) + b_1(h + \delta(h))}{\alpha(h) - b_1(h + \delta(h))} \int\limits_0^t |v(\tau)|^2\, d\tau + \frac{|(\xi_0^h - x_0)'D(\xi_0^h - x_0)|}{\alpha(h) - b_1(h + \delta(h))}, \qquad t \in T. \tag{54}$$

The estimate (17) (in the second case) follows from inequality (54).

Let us verify inequality (18). By using the estimates (45) and (50) and by taking into account Condition 3, by analogy with (51), we obtain the inequality

$$\dot{V}(t)|_{(44)} = \frac{d}{dt}|z'(t)Dz(t)| \le \chi|z(t)|^2 + 2(D(\psi_i^h - \xi_i^h))'C(v^h(t) - v(t)) \\ + b_1(h + \delta(h))\{|v^h(t)| + |v(t)|\}.$$

The last inequality, together with (41), implies that

$$\dot{V}(t)|_{(44)} \le \chi|z(t)|^2 + 2c_0^2\alpha(h) + 2c_0 b_1(h + \delta(h)).$$

By taking into account (20), from this, we derive the relation

$$\frac{d}{dt}|z'(t)Dz(t)| = \chi_1|z'(t)Dz(t)| + 2c_0^2\alpha(h) + 2c_0 b_1(h + \delta(h)) + \psi_1(t) \quad (\text{almost all} \quad t \ge 0),$$

where $\psi_1(t) \le 0$ $(t \ge 0)$. By using the last relation, by analogy with (42), we have

$$\chi_*|z(t)|^2 \le |z'(t)Dz(t)|^2 \le |z'(0)Dz(0)|e^{\chi_1 t} + \int_0^t e^{\chi_1(t-\tau)}\{2c_0^2\alpha(h) + 2c_0 b_1(h+\delta(h))\}\,d\tau$$

$$\le |z'(0)Dz(0)|e^{\chi_1 t} - \frac{2c_0^2\alpha(h) + 2c_0 b_1(h+\delta(h))}{\chi_1}, \qquad t \in T.$$

This implies inequality (18) (in the second case). The proof of the theorem is complete.

From Theorem 1, we obtain the main assertion that provides the solution of the above-posed problem on the rough inversion of system (1).

Theorem 2. *Let the assumptions of Theorem* 1 *hold. Then the family* $(V_h(\cdot\,,\cdot))_{h>0}$ *of admissible feedbacks of the form* (16) *is roughly inverting, and the pair* $(\gamma_1(\cdot), \gamma_2(\cdot))$, *where*

$$\gamma_1(h) = \nu(h, \alpha(h)), \qquad \gamma_2(h) = \varrho_2(h) \qquad (h > 0),$$

is an estimate of the accuracy of that family.

Proof. Note first that $\gamma_1(h), \gamma_2(h) \to 0$ as $h \to 0$. Let $h > 0$, and let $(x(\cdot), \xi^h(\cdot), y^h(\cdot), v^h(\cdot))$ be a control process corresponding to an admissible feedback $V_h(\cdot\,,\cdot)$, an admissible perturbation $v(\cdot)$, and measurement accuracy h. Then $|\xi_0^h - x_0| \le h$ and, by Theorem 1, inequalities (17) and (18) are true for all $t \ge 0$, which implies the assertion of the theorem. The proof of the theorem is complete.

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REFERENCES

1. Kryazhimskii, A.V. and Osipov, Yu.S., Modeling of a Control in a Dynamical System, *Izv. Akad. Nauk SSSR Tekhn. Kibern.*, 1983, no. 2, pp. 51–60.
2. Krasovskii, N.N. and Subbotin, A.I., *Pozitsionnye differentsial'nye igry* (Positional Differential Games), Moscow: Nauka, 1974.
3. Tikhonov, A.N. and Arsenin, V.Ya., *Metody resheniya nekorrektnykh zadach* (Methods for Solving Ill-Posed Problems), Moscow: Nauka, 1979.
4. Osipov, Yu.S. and Kryazhimskii, A.V., *Inverse Problems for Ordinary Differential Equations: Dynamical Solutions*, Basel, 1995.
5. Osipov, Yu.S., Kryazhimskii, A.V., and Maksimov, V.I., *Metody dinamicheskogo vosstanovleniya vkhodov upravlyaemykh sistem* (Methods of Dynamic Reconstruction of Control Systems), Ekaterinburg, 2011.
6. Maksimov, V.I., The Method of Lyapunov Functions in Problems of Reconstruction of Inputs in Systems with Aftereffect, *Sovrem. Mat. Prilozh. Nelin. Din.*, 2005, vol. 26, pp. 78–95.
7. Osipov, Yu.S., Kryazhimskii, A.V., and Maksimov, V.I., Dynamic Inverse Problems for Parabolic Systems, *Differ. Uravn.*, 2000, vol. 36, no. 5, pp. 579–597.
8. Maksimov, V.I., *Zadachi dinamicheskogo vosstanovleniya vkhodov beskonechnomernykh sistem* (Problems of Dynamic Reconstruction of Inputs of Infinite-Dimensional Systems), Yekaterinburg, 2000.
9. Kryazhimskii, A.V. and Maksimov, V.I., On the Stabilization of Dynamical Systems in the Case of Incomplete Information, *Preprint IMM*, Yekaterinburg, 2004.
10. Kryazhimskii, A.V. and Maksimov, V.I., On Rough Inversion of a Dynamical System with a Disturbance, *J. Inverse Ill-Posed Probl.*, 2008, vol. 16, pp. 1–14.
11. Kryazhimskii, A.V. and Maksimov, V.I., Resource-Saving Infinite-Horizon Tracing under Uncertain Input, *Appl. Math. Comput.*, 2010, vol. 217, pp. 1135–1140.
12. Barbashin, E.A., *Vvedenie v teoriyu ustoichivosti* (Introduction to the Theory of Stability), Moscow: Nauka, 1967.